CALCULUS OF VARIATION

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Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line x+y=4.

Solution:

If S is the length of the arc of the curve y=f(x) connecting the point (x_0, y_0) and (x_1, y_1).

Then S=$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$

To find the minimum value of the functional S, where the two points move along $\boldsymbol{x^2 + y^2} = 1$ and x+y=4

To find the extremum value of the integral

I=$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$

If I is minimum, then $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, where

F=$\sqrt{1 + y'^2}$

$$\Rightarrow -\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = k \quad \Rightarrow y'^2 = \frac{k^2}{1-k^2}$$

- $\therefore y' = \frac{k}{\sqrt{1-k^2}} = c_1$
- $y = c_1 x + c_2 \qquad \rightarrow ①$
- Let $\Phi(x) = x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-x^2}$

$\psi(x) = x + y = 4 \Rightarrow y = 4 - x$

$① \Rightarrow \sqrt{1 - {x_0}^2} = c_1 x_0 + c_2 \rightarrow ②$

$4 - x_1 = c_1 x_1 + c_2 \rightarrow ③$

The transversality condition is

$$\left[F + (\Phi' - y')\ \frac{\partial F}{\partial y'}\right]_{(x_0, y_0)} = 0$$

$$\left(\sqrt{1 + y'^2} + \left(\frac{-x_0}{\sqrt{1-{x_0}^2}} - y'\right) \frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow 1 + y'^2 - \frac{x_0 y'}{\sqrt{1-{x_0}^2}} - y'^2 = 0$$

$\Rightarrow x_0 y' = \sqrt{1 - x_0^2}$

$\Rightarrow \boldsymbol{x_0 c_1 = \sqrt{1 - x_0^2}}$

$\Rightarrow \boldsymbol{x_0^2 (c_1^2 + 1) = 1} \rightarrow ④$

The transversality condition is

$$\left[F + (\psi' - y') \frac{\partial F}{\partial y'}\right]_{(x_1, y_1)} = 0$$

$$\Rightarrow \sqrt{1 + y'^2} + (-1 - y') \frac{y'}{\sqrt{1 + y'^2}} = 0$$

$\Rightarrow y'_1 = 1$

$\Rightarrow \boldsymbol{c_1} = 1 \rightarrow ⑤$

Sub ⑤ in ④, $\boldsymbol{x_0^2 (1 + 1) = 1}$

- $\Rightarrow x_0 = \frac{1}{\sqrt{2}} \rightarrow ⑥$
- Sub ⑥ and ⑤ in ②, $\sqrt{1 - {x_0}^2} = c_1 x_0 + c_2$
- $c_2 = \frac{1}{\sqrt{2}} - \sqrt{1 - \frac{1}{2}} = 0$
- Sub c_2 and c_1 in ③ 4-$x_1 = c_1 x_1 + c_2$
- 4-$x_1 = x_1$
- $\therefore x_1 = 2$
- The shortest distance
- $I = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$

$I = \int_{\frac{1}{\sqrt{2}}}^{2} \sqrt{1 + 1^2}\, dx$

$= \sqrt{2}\left(2 - \frac{1}{\sqrt{2}}\right)$

$I = 2\sqrt{2} - 1$

- 2) Test for an extremum of the functional
- $\int_0^{x_1} \sqrt{\frac{1+y'^2}{y}} dx$ given that y(0)=0 and $y_1 = x_1 - 5$
- Solution:
- Now, $F = \sqrt{\frac{1+y'^2}{y}}$
- Since F is independent of x,
- The Eulers Equation is given by,
- $\frac{d}{dx}\left[F - y' \frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0 \rightarrow (1)$
- $\frac{\partial F}{\partial y'} = \frac{y'}{y\sqrt{1+y'^2}}$
- $(1) \rightarrow \frac{d}{dx}\left[\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}}\right] = 0$

- $\left[\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}}\right]$=a(constant)
- $\frac{1}{y\sqrt{1+y'^2}}$=a
- $y'^2 = \frac{1-y^2a^2}{y^2a^2}$
- $\frac{dy}{dx} = \frac{\sqrt{1+y'^2a^2}}{ay}$
- $dx = \frac{aydy}{\sqrt{1+y'^2a^2}}$
- Integrating,
- a$\int \frac{ydy}{\sqrt{1+y'^2a^2}}$=x+b
- Put, $1-y^2a^2=t^2$
- ydy=$\frac{-tdt}{a^2}$

- a$\int \frac{-tdt}{a^2 t}$=x+b
- $\frac{-1}{a}$t=x+b
- $\frac{-1}{a}\sqrt{1-a^2y^2}$=x+b
- Squaring,
- $\frac{1}{a^2}-y^2=(x+b)^2$
- Take, $\frac{1}{a^2}=k^2$
- $(x+b)^2+y^2=k^2 \rightarrow (1)$
- By the transversality condition,
- F+(Ø′−y’)$\frac{\partial F}{\partial y\prime}$=0 y=Ø$(x)$
- y=x−5 and so $\emptyset = x-5$
- $\emptyset' = 1$
- $\frac{\sqrt{1+y\prime^2}}{y}$+(1−y’)$\frac{y\prime}{y\sqrt{1+y\prime^2}}$=0

- $\frac{1+y'^2+y'-y'^2}{y\sqrt{1+y'^2}}=0$
- $1+y'=0$
- $y'=-1$
- Integrating,
- $y=-x-b$
- $y=-(x+b)$
- $y^2=(x+b)^2$
- $(x-5)^2=(x+b)^2$
- $x^2-2(5)(x)+25=x^2+2(b)(x)+b^2$
- $-2(5)x=2bx$
- $b=-5$

- Given: y(0)=0
- $(x + b)^2 - k^2 = -y^2$
- $b^2 - k^2 = 0$
- $k^2 = 25$
- Sub these values in (1),
- $(x - 5)^2 + y^2 = 25$